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AN APPLICATION OF ELLIPTIC FUNCTIONS TO A PROBLEM IN GEOMETRY.

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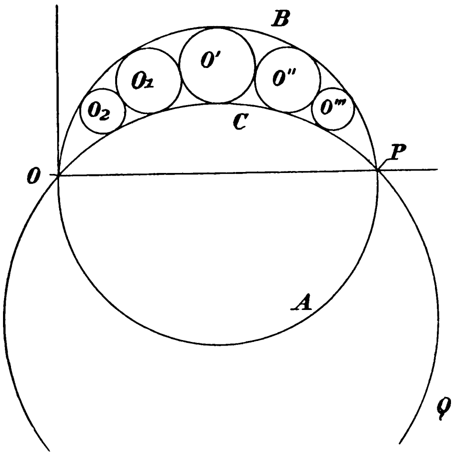
In one of the crescents formed by two intersecting circles, smaller circles are inscribed touching each other successively. Required (1) the sum of the radii of the inscribed circles; (2) the sum of their areas.

I.

The circles OBP and OCQ (Fig. 1), or

$$\left.\begin{aligned} x^2+y^2+bx+cy&=0\\ x^2+y^2+b'x+c'y&=0\end{aligned}\right\}, \tag{1}$$

Fig. 1.



after reciprocation by means of the transformation

$$x=\frac{k^2X}{X^2+Y^2}\text{ and }y=\frac{k^2Y}{X^2+Y^2}, \tag{2}$$

O being the centre of reciprocation, transform into the straight lines (Fig. 2), OST and $OS'T'$, or

$$\left.\begin{aligned} b\,X+c\,Y+k^2&=0\\ b'X+c'Y+k^2&=0\end{aligned}\right\},$$

which make the same angle with each other as the two circles OAP and OCQ.

The circles . . . , O_2, O_1, O', O'', O''', . . . of Fig. 1 are transformed into the circles . . . , O_1, O', O'', . . . of Fig. 2, touching the straight lines OST and $OS'T'$. Conversely any two straight lines OST and $OS'T'$ inclosing the same angle that any two given intersecting circles do may be transformed into the latter by a proper choice of the centre of reciprocation.

We therefore attack the given problem as follows: Draw the lines OST, $OS'T'$ (Fig. 2) making with each other an angle equal to that formed by the given circles (Fig. 1).

Let z^2 be the radius of any arbitrary circle inscribed in the angle $TSOS'T'$ and let OO' be the bisector of this angle. Call the angle TOO', α. Then

$$TO'' = z^2\frac{1+\sin\alpha}{1-\sin\alpha} = z^2h^{-2},$$

where h^2 is put for $\frac{1-\sin\alpha}{1+\sin\alpha}$ (which is < 1).

The radii of the series of inscribed circles . . . , O_1, O', O'', . . . will therefore be

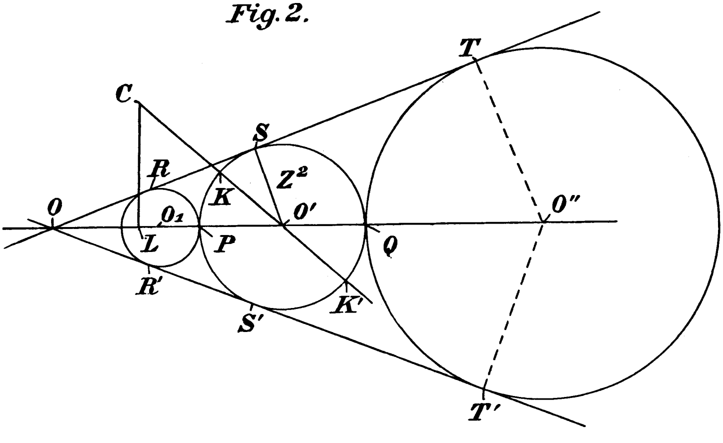
$$\ldots,\ h^4z^2,\ h^2z^2,\ z^2,\ h^{-2}z^2,\ h^{-4}z^2,\ \ldots. \tag{3}$$

In Fig. 2, from any point C draw CL perpendicular to OO', meeting OO' in some point L. Put $OL = a$ and $CL = b$. Draw a straight line through C and O' meeting the circumference of O' in K and K'.

Since

$$CK = \sqrt{b^2 + (z^2\operatorname{cosec}\alpha - a)^2} - z^2,$$

Fig. 2.



and

$$CK' = \sqrt{b^2 + (z^2 \operatorname{cosec} a - a)^2} + z^2,$$

therefore, after reciprocation with respect to the centre C, the radius of the circle in the crescent (Fig. 1) which corresponds to the circle O' of radius z^2, will be equal to

$$\tfrac{1}{2}\left[\frac{1}{\sqrt{b^2 + (z^2 \operatorname{cosec} a - a)^2} - z^2} - \frac{1}{\sqrt{b^2 + (z^2 \operatorname{cosec} a - a)^2} + z^2}\right]$$

or

$$\frac{\tan^2 a \,.\, z^2}{(z^2 - z_1^{\,2})\,(z^2 - z_2^{\,2})}, \tag{4}$$

where $z_1^{\,2}$ and $z_2^{\,2}$ are the roots of the equation

$$(z^2)^2 - \frac{2a}{\cos a}(z^2) + (a^2 + b^2)\tan^2 a = 0,$$

considered as a quadratic equation.

In this equation a is given, being one-half the angle which the two given circles make with each other. a and b are also known since they are determined by the condition that the centre C (Fig. 2) of reciprocation was so chosen that the two straight lines OST and $OS'T'$ transform into the two circles OCQ and $OAPB$ (Fig. 1).

Hence the radius of the reciprocal of any nth circle will be

$$\frac{\tan^2 a \,.\, h^{\pm 2n} z^2}{(h^{\pm 2n} z^2 - z_1^{\,2})\,(h^{\pm 2n} z^2 - z_2^{\,2})}.$$

Therefore the sums of the radii and areas of the circles in the crescent will be respectively,

$$S_r(z) = \sum_{+\infty}^{-\infty}{}_n \frac{\tan^2 a \,.\, h^{2n}\, z^2}{(h^{2n}\, z^2 - z_1^{\,2})\,(h^{2n}\, z^2 - z_2^{\,2})}, \tag{5}$$

$$S_A(z) = \sum_{+\infty}^{-\infty}{}_n \frac{\pi \tan^4 a \,.\, h^{4n}\, z^4}{(h^{2n}\, z^2 - z_1^{\,2})^2\,(h^{2n}\, z^2 - z_2^{\,2})^2}. \tag{6}$$

II.

In this section we shall calculate $S_r(z)$ in terms of σ-functions.

From § 8, (6) of Weierstrass and Schwarz's Elliptic Function Formulas (Edition 1885), we have

$$\frac{\sigma'}{\sigma}(u) = \frac{\eta}{\omega}u + \frac{\pi i}{2\omega}\left[\frac{z^2+1}{z^2-1} + \sum{}_n \frac{2h^{2n}\, z^{-2}}{1 - h^{2n}\, z^{-2}} - \sum{}_n \frac{2h^{2n}\, z^2}{1 - h^{2n}\, z^2}\right], \tag{7}$$

where, according to Prof. Schwarz's notation,

$$z = e^{\frac{u\pi i}{2\omega}}, \quad h = e^{\frac{\omega'\pi i}{\omega}};$$

and we put, analogously,

$$z_1 = e^{\frac{u_1\pi i}{2\omega}} \text{ and } z_2 = e^{\frac{u_2\pi i}{2\omega}};$$

whence

$$\frac{z}{z_1} = e^{\frac{(u-u_1)\pi i}{2\omega}} \text{ and } \frac{z}{z_2} = e^{\frac{(u-u_2)\pi i}{2\omega}};$$

that is, when u becomes $u - u_1$ or $u - u_2$, z becomes $\frac{z}{z_1}$ or $\frac{z}{z_2}$ respectively.

Hence,

$$\frac{\sigma'}{\sigma}(u-u_1) = \frac{\eta}{\omega}(u-u_1) + \frac{\pi i}{2\omega}\left[\frac{z^2+z_1^2}{z^2-z_1^2} + \sum^n \frac{2h^{2n}z^{-2}}{z_1^{-2} - h^{2n}z^{-2}} - \sum^n \frac{2h^{2n}z^2}{z_1^2 - h^{2n}z^2}\right], \tag{8}$$

$$\frac{\sigma'}{\sigma}(u-u_2) = \frac{\eta}{\omega}(u-u_2) + \frac{\pi i}{2\omega}\left[\frac{z^2+z_2^2}{z^2-z_2^2} + \sum^n \frac{2h^{2n}z^{-2}}{z_2^{-2} - h^{2n}z^{-2}} - \sum^n \frac{2h^{2n}z^2}{z_2^2 - h^{2n}z^2}\right]. \tag{9}$$

Subtracting (9) from (8)

$$\frac{\sigma'}{\sigma}(u-u_1) - \frac{\sigma'}{\sigma}(u-u_2) = -\frac{\eta}{\omega}(u_1-u_2) + \frac{\pi i\,(z_1^2 - z_2^2)}{\omega}\sum_{+\infty}^{-\infty}{}^{n} \frac{h^{2n}z^2}{(h^{2n}z^2 - z_1^2)(h^{2n}z^2 - z_2^2)}.$$

After transposing $-\frac{\eta}{\omega}(u_1 - u_2)$ and dividing by $\frac{\pi i(z_1^2 - z_2^2)}{\omega \tan^2\alpha}$ we obtain

$$S_r(z) = \sum_{+\infty}^{-\infty}{}^{n} \frac{\tan^2\alpha \,.\, z^2 h^{2n}}{(h^{2n}z^2 - z_1^2)(h^{2n}z^2 - z_2^2)}$$

$$= \frac{\omega\tan^2\alpha}{\pi i\,(z_1^2 - z_2^2)}\left[\frac{\sigma'}{\sigma}(u-u_1) - \frac{\sigma'}{\sigma}(u-u_2)\right] + \frac{\eta\,(u_1-u_2)}{\pi i\,(z_1^2 - z_2^2)}\tan^2\alpha. \tag{10}$$

Hence $S_r(z)$ is determined, being expressed in terms of known σ and σ' functions and of given functions of known constants. For the period ω depends on the choice of the first arbitrary circle; 2α is the angle at which the two given circles intersect; z_1 and z_2 are the roots of a given quadratic equation (see quadratic equation under (4)); u, u_1, u_2, and ω' are respectively determined by the equations

$$z = e^{\frac{u\pi i}{2\omega}}, \qquad z_1 = e^{\frac{u_1\pi i}{2\omega}}, \qquad z_2 = e^{\frac{u_2\pi i}{2\omega}},$$

$$\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}} = h = e^{\frac{\omega'}{\omega}\pi i};$$

and the function η is determined by the equation

$$\eta = \frac{\pi^2}{2\omega}\left[\frac{1}{6} - \sum^n \frac{4h^{2n}}{(1-h^{2n})^2}\right].$$

See Weierstrass and Schwarz's Elliptic Function Formulas, § 6, (10).

The h introduced under (3) is compatible with h in (7); since in the Σ's (7) and those that follow, that they may be convergent, h must be < 1. (See Weierstrass and Schwarz, § 6, (3)). This restriction we have expressed under (2).

III.

Here it is only intended to suggest a plan, similar to that employed in I., by which the function $S_A(z)$ may be calculated. This may be done as follows: By making use of formulas (6), § 8, Weierstrass and Schwarz's Elliptic Function Formulas, also of (4), § 9, and taking note of the fact, as above, that when u becomes $u - u_1$, or $u - u_2$, z becomes z/z_1 or z/z_2 respectively, one can construct the function $S_A(z)$, which will have the form

$$C_1\,\mathrm{p}\,(u-u_1) + C_2\,\mathrm{p}\,(u-u_2) + C_3\frac{\sigma'}{\sigma}(u-u_1) + C_4\frac{\sigma'}{\sigma}(u-u_2) + C_5 = S_A(z). \qquad (11)$$

The C's being constants which are functions of known constants, z_1, z_2, u_1, u_2, etc.

GÖTTINGEN, *July* 3, 1891.